9. H. Honji and M. Tatsuno, "Vortex rings in a stratified fluid," J. Phys. Soc. Jpn., 41, No. 6 (1976).
10. H.-P. Pao and T. Kao, "Vortex structure in the wake of a sphere," Phys. Fluids, 20, No. 2 (1977).

STABILITY OF VISCOUS WALL JET

O. S. Ryzhov

UDC 533.6.12

An incompressible, viscous plane jet along a rigid wall is considered. The theory of free interactions between the boundary layer and the external potential flow is used to study its stability. The dispersion equation relating the frequency of free oscillations to the wave number is identical to the equation that governs the stability of Poiseuille flow in an infinite channel. The properties of the solution to the problem of harmonic disturbances generated by the oscillator used in the test setup depend on the location of the roots of the dispersion equation. It is observed that the analysis of the temporal amplification of disturbances can be carried out using Prandtl's boundary layer equations with self-induced pressure.

1. Consider an incompressible, viscous plane jet along a rigid wall. The entire jet thickness can be treated as a boundary layer whose dimensionless velocity distribution U_1 at any section is shown in Fig. 1. It is significant that the velocity, as well as its derivative with respect to the normal coordinate Y_1, are zero at the outer edge of the boundary layer. Such velocity profiles are typical not only for jets; as is well known, similar velocity profiles are obtained in steady flow on a heated vertical plate [1] and on a rotating disk [2]. The theory of free interactions between the boundary layer with the external potential flow has been used to study the characteristics of these flows near the leading and trailing edges of solid bodies [3]. More examples are considered in [4], in which a solution of Prandtl's equation has been obtained to describe the separation of the jet and the subsequent development of the recirculating zone. This theory is applied to analyze the stability of the jet with respect to large wavelength disturbances, with the critical layer of neutral disturbances close to the wall [5, 6]. These disturbances determine, in the linear approximation, the asymptote of curves relating wave number to Reynolds number, as the latter goes to infinity [7].

The whole velocity field is divided into two regions to achieve this objective. According to the principles of free interaction theory [8, 9], the effect of viscosity on the structure of the disturbed flow in the upper region 1 is negligibly small. Let us introduce a small parameter $\varepsilon = \mathrm{Re}^{-1/4}$, where the Reynolds number $\mathrm{Re} = \rho^* U_M^* L^*/\lambda^*$ is expressed in terms of density ρ^*, the coefficient of viscosity λ^*, the maximum U_M^* in the jet, and its characteristic length L^*. The time t^*, and cartesian coordinates x^*, y^* are given by the following equations:

$$t^* = \frac{L^*}{U_m^*}(t_0 + \varepsilon^4 t_1),\ x^* = L^*(x_0 + \varepsilon^6 x_1),\ y^* = \varepsilon^7 L^* y_1, \tag{1.1}$$

where t_0 and x_0 are arbitrary constants. The pressure p^* and the velocity components v_x^*, v_y^* are expressed by the asymptotic series:

$$p^* = p_\infty^* + \rho^* U_m^{*2}[\varepsilon^4 p_1(t_1, x_1, y_1) + \dots],$$
$$v_x^* = U_m^*[U_1(y_1) + \varepsilon^2 u_1(t_1, x_1, y_1) + \dots],\ v_y^* = U_m^*[\varepsilon^3 v_1(t_1, x_1, y_1) + \dots], \tag{1.2}$$

where p_∞^* is the pressure at the outer edge of the jet and the quantity $U_1(Y_1)$ is the velocity profile in the initial undisturbed flow.

Equations (1.1) and (1.2) are substituted into the Navier–Stokes equations and only the principal terms in the resulting equations are retained; we obtained [3, 4]

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 26-33, March-April, 1982. Original article submitted February 19, 1981.

0021-8944/82/2302-0181$07.50

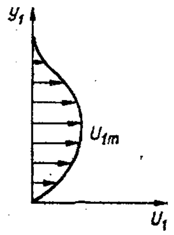

Fig. 1

$$p_1 = P(t_1, x_1) + \frac{\partial^2 A_2}{\partial x_1^2}\int_0^{y_1} U_1^2(Y_1)\,dY_1,\; u_1 = A_2(t_1, x_1)\frac{dU_1}{dy_1},$$
$$v_1 = -U_1(y_1)\frac{\partial A_2}{\partial x_1}. \tag{1.3}$$

Here the function $P(t_1, x_1)$ is the pressure disturbance along the rigid wall, and the function $A_2(t_1, x_1)$ is the instantaneous displacement of the streamline. Both these functions are arbitrary. Periodicity along the streamwise coordinate x_1 is superimposed on them to study the flow stability. The problem analyzed is quasisteady: time plays the role of a parameter [10-12].

Disturbances should decay outside the boundary layer. In view of the above assumptions ($U_1 \to 0$, $dU_1/dY_1 \to 0$ as $y_1 \to \infty$), the velocity components u_1, v_1 automatically satisfy this requirement. The condition of the pressure disturbance p_1 going to zero leads to the equation $P = -\Delta\frac{\partial^2 A_2}{\partial x_1^2}$, $\Delta = \int_0^\infty U_1^2(Y_1)\,dY_1$, which relates the functions P and A_2. The numerical value of Δ depends on the velocity distribution across the boundary layer in the basic problem.

2. The fluid viscosity in the thin wall region 2 has a critical influence on the disturbance structure. Here, the independent variables are

$$t^* = \frac{L^*}{U_m^*}(t_0 + \varepsilon^4 t_2),\; x^* = L^*(x_0 + \varepsilon^6 x_2),\; y^* = \varepsilon^9 L^* y_2, \tag{2.1}$$

and the required functions are

$$p^* = p_\infty^* + \rho^* U_m^{*2}[\varepsilon^4 p_2(t_2, x_2, y_2) + \ldots],$$
$$v_x^* = U_m^*[\varepsilon^2 u_2(t_2, x_2, y_2) + \ldots],\; v_y^* = U_m^*[\varepsilon^5 v_2(t_2, x_2, y_2) + \ldots]. \tag{2.2}$$

Following the usual procedure for the free interaction problem [8, 9], the flow is assumed to have a layered structure and hence the scales for both time and the streamwise coordinate are chosen identical in both regions but their thickness is evaluated independently. Hence $t_1 = t_2$, $x_1 = x_2$, but $y_1 \neq y_2$. The substitution of Eqs. (2.1), (2.2) into the Navier–Stokes equations results in the following set of equations for the principal terms:

$$\frac{\partial u_2}{\partial x_2} + \frac{\partial v_2}{\partial y_2} = 0,\; \frac{\partial p_2}{\partial y_2} = 0,\; \frac{\partial u_2}{\partial t_2} + u_2\frac{\partial u_2}{\partial x_2} + v_2\frac{\partial u_2}{\partial y_2} = -\frac{\partial p_2}{\partial x_2} + \frac{\partial^2 u_2}{\partial y_2^2}. \tag{2.3}$$

These are Prandtl's boundary layer equations, though the pressure p_2 in them is not specified *a priori* but is determined as a result of the solution of the complete problem. In fact, matching with the series expansion for the outer region, in which p_1 is expressed in terms of the first equation in (1.3) and Eq. (1.4), leads to the expression

$$p_2 = -\Delta\partial^2 A_2/\partial x_2^2. \tag{2.4}$$

Besides, matching the principal terms for the streamwise velocity components leads to the limiting condition

$$u_2 - \lambda y_2 \to \lambda A_2(t_2, x_2) \text{ for } y_2 \to \infty \tag{2.5}$$

with the constant $\lambda = dU_1(0)/dy_1$. The remaining boundary conditions for the system of Eqs. (2.3), (2.4) are expressed in terms of the requirement of periodicity of the functions along the streamwise coordinate x_1 and the equations u_2 and $v_2 = 0$ at $y_2 = 0$ that ensure no slip conditions at the solid surface. It must be observed that according to Eq. (2.4), the pressure disturbance is proportional to the streamline curvature [3, 4] and not to the integral of their gradients as in the case of the boundary layer on a body in a uniform flow [9].

It is easy to observe that the system of equations (2.3) is invariant to the similarity transformation

$$t_2 = \Delta^{2/7}\lambda^{-8/7}t,\ x_2 = \Delta^{3/7}\lambda^{-5/7}x,\ y_2 = \Delta^{1/7}\lambda^{-4/7}y,$$
$$p_2 = \Delta^{2/7}\lambda^{6/7}p,\ A_2 = \Delta^{1/7}\lambda^{-4/7}A,\ u_2 = \Delta^{1/7}\lambda^{3/7}u, \tag{2.6}$$
$$v_2 = \Delta^{-1/7}\lambda^{4/7}v,$$

the coefficient Δ is eliminated from Eq. (2.4) using this transformation, whereas the constant λ is eliminated using the limiting condition in Eq. (2.5). In what follows we shall consider the transformation (2.6) satisfied and drop the subscript 2 from both the independent variables and the unknown functions. Using a new set of reference quantities it is necessary to put $\Delta = \lambda = 1$ in Eqs. (2.4), (2.5).

3. Following the usual procedure for stability theory [7], the solution describing free fluctuations of viscous fluid is sought in the form

$$p = -k^2A = -ak^2e^{\omega t+kx},\ u = y - ae^{\omega t+kx}\frac{df}{dy},\ v = ake^{\omega t+kx}f(y). \tag{3.1}$$

Series (3.1) is substituted into the system of equations (2.3), (2.4), and the resulting equations are linearized with respect to the amplitude of disturbances a. The following is the resulting third-order ordinary differential equation for the function f:

$$d^3f/dy^3 - (\omega + ky)df/dy + kf - k^3 = 0, \tag{3.2}$$

which is conveniently analyzed [13] in the complex plane $z = \omega k^{-2/3} + k^{1/3}y$. In order to separate the single-valued branch of the function $k^{1/3}$, we introduce a cut in the complex plane k along the negative real axis and set $\pi \leqslant \arg k \leqslant \pi$. Differentiating (3.2) with respect to y and switching over to the variable z, we get Airy's equation for the second derivative d^2f/dz^2. Then it is easy to write the solution for the function f itself which satisfies the no-slip condition at the plate surface $y = 0$, i.e.,

$$f = c\int_{\zeta}^{z} dz'' \int_{\zeta}^{z''} \mathrm{Ai}(z')\,dz',\ \zeta = \frac{\omega}{k^{2/3}}. \tag{3.3}$$

Here Ai(z') is the Airy function; c is an arbitrary constant. Substitution of Eq. (3.3) into the basic Eq. (3.2) results in the expression for the constant $c = k^2\left[\frac{d\mathrm{Ai}(\zeta)}{dz}\right]^{-1}$.

The limiting condition at the outer edge of the wall region remains to be satisfied. As seen from Eq. (2.5), $df/dy \to -1$ as $y \to \infty$. This is used to derive the dispersion equation

$$\frac{d\mathrm{Ai}(\zeta)}{dz}\left[\int_{\zeta}^{\infty} \mathrm{Ai}(z)\,dz\right]^{-1} = Q = -k^{7/3}, \tag{3.4}$$

relating the frequency of free oscillations to the wave number. It is possible to verify that there is an identical expression for the dispersion relation which determines the stability of Poiseuille flow in a plane channel with respect to large wavelength antisymmetric disturbances, with the critical layer of neutral disturbances in the neighborhood of the wall. The fact that it takes place within a narrow layer has a critical effect on the characteristics of flow stability. In the first case, the jet is bounded by a stationary region at the top and in the second case the upper boundary is the rigid wall. However, the dispersion equation is identical and independent of the type of these boundaries and the nature of the fluid flow.

It is immediately possible to indicate two properties of the solutions which enable us to judge the loss of stability of the given flow. In order to do this, we use the well-known results of studies on free interaction of boundary layer on a flat plate placed in an infinite uniform flow of an incompressible fluid [5, 6].

First, there is an infinite set of roots located in the neighborhood of the negative real axis corresponding to the given value of k (or ω) in the complex plane ζ. Let arg $\zeta = \vartheta = \pi \mp \vartheta'$, arg k = ϑ_k. Following [13], we put $|\zeta| \to \infty$ as $\vartheta' |\zeta|^{3/2} \to 0$. We use the asymptotic expression $\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi} z^{1/4}} \left[\exp\left(-\frac{2}{3} z^{3/2}\right) + i \exp\left(\frac{2}{3} z^{3/2}\right)\right]$ for the Airy function, which remains continuous across the negative real axis, to simplify the dispersion Eq. (3.4). After a few simple transformations, the latter results in two real equations:

$$|\zeta|^{1/4} \cos\left(\frac{2}{3} |\zeta|^{3/2} + \frac{\pi}{4}\right) = \sqrt{\pi} |k|^{7/3} \cos \frac{7}{3} \vartheta_k,$$

$$\vartheta' |\zeta|^{7/4} \sin\left(\frac{2}{3} |\zeta|^{3/2} + \frac{\pi}{4}\right) = -\sqrt{\pi} |k|^{7/3} \sin \frac{7}{3} \vartheta_k,$$

from which the statement made above follows directly. In particular, when $\vartheta_k = (3/7)\pi$, roots of the dispersion equation with large numbers in the complex plane ζ are found on the negative real axis.

Secondly, of all the roots with purely imaginary values of k (playing the fundamental role in the study of viscous flow stability), only in one case can the real part of ω take negative as well as positive values. As regards the real part of ω of all the remaining roots, it is less than zero, if $|k| \neq 0$. The particular case with the real part of ω equal to zero for the first root ζ_1 corresponds to the Tollmien–Schlichting traveling waves in which there are neutral disturbances having a constant amplitude with time. All roots with imaginary values of k are found by a simple conversion of similar solutions from the boundary layer-inviscid free interaction theory for a semiinfinite flat plate [5, 6] when the quantity $Q = \mp ik^{4/3}$ in the right-hand side of the dispersion Eq. (3.9), the upper sign being taken for Im k > 0, and the lower for Im k < 0. Actually, both for the jet and the boundary layer layer on a semiinfinite flat plate only $|Q|$ is different whereas arg Q = $\pm\pi/6$, the signs being chosen as before. Since $\zeta = \omega/k^{2/3}$, arg ω is also identical for the disturbances propagated in these flows. Using the results given in [5, 6], the absolute values of the critical wave number k_* is immediately found for the neutral disturbances of the fluid jet, with Re ω = 0 for the first root ζ_1; i.e., $k_* = 1.0003$.

4. Consider forced oscillations of the jet generated by an oscillator setup on the wall. Let it be expressed by the equation $y = a \exp(i\Omega t) h(x)$, with the function h different from zero only in the interval $0 \leqslant x \leqslant l$ Boundary conditions on the vibration wall take the form

$$u = 0, \; v = ia\Omega \, e^{i\Omega t} h(x). \tag{4.1}$$

As before, assume the amplitude parameter $a \ll 1$ in the boundary conditions (4.1) and linearize the equations of fluid motion with respect to this parameter. After eliminating the time-dependent quantities from the unknown disturbance functions with the help of the equations

$$p = ae^{i\Omega t} p'(x), \; A = ae^{i\Omega t} A'(x),$$
$$u = y + ae^{i\Omega t} u'(x, y), \; v = ae^{i\Omega t} v'(x, y), \tag{4.2}$$

the terms depending on spatial coordinates are expressed in terms of Fourier integrals. Equation (2.4) is used to write

$$\bar{p}(K) = K^2 \bar{A}(K) = K^2 \int_{-\infty}^{\infty} e^{-iKx} A'(x) \, dx,$$

$$\bar{u}(K, y) = \int_{-\infty}^{\infty} e^{-iKx} u'(x, y) \, dx, \; v(K, y) = \int_{-\infty}^{\infty} e^{-iKx} v'(x, y) \, dx. \tag{4.3}$$

The function F(K, Y) is introduced as

$$\bar{u} = -dF/dy, \; \bar{v} = iKF. \tag{4.4}$$

The substitution of Eqs. (4.2)-(4.4) in the equations of motion gives

$$\frac{d^3F}{dy^3} - i(\Omega + Ky)\frac{dF}{dy} + iKF + iK^3\overline{A} = 0. \tag{4.5}$$

This linear differential equation differs from (3.2) only by the value of the free parameter; the coefficients of both equations, when $\Omega = -i\omega$ and $K = ik$, are, of course, identical. It is clear from the above that in the complex plane $Z = i^{1/3}(\Omega K^{-2/3} + K^{1/3}y)$ it leads to the Airy equation for the second derivative d^2F/dZ^2. The boundary conditions follow from (4.1)

$$F = (\Omega/K)\overline{h},\ dF/dy = \overline{h} \quad \text{at} \quad y = 0$$

for Eq. (4.5), where $\overline{h}(k)$ is the Fourier transform of the function $h(x)$, giving the form of the oscillator. They allow us to conclude that in the complex plane Z, the third derivative is given by

$$d^3F/dZ^3 = -K^2\overline{A} \quad \text{with} \quad Z = Z^0 = i^{1/3}\Omega K^{-2/3}. \tag{4.6}$$

The limiting condition (2.5) gives the requirement

$$dF/dZ \to -(iK)^{-1/3}\overline{A} \quad \text{at}\ |Z| \to \infty. \tag{4.7}$$

The solution of Airy equations satisfying the boundary conditions (4.6) and damping at infinity has the form

$$\frac{d^2F}{dZ^2} = -K^2\overline{A}\left[\frac{d\mathrm{Ai}(Z^0)}{dZ}\right]^{-1}\mathrm{Ai}(Z).$$

The last equation is integrated once, using the boundary conditions on the vibrating wall. The substitution of the result of integration in the limiting conditions (4.7) determines the Fourier transform

$$\overline{A} = -\frac{\overline{h}(K)\,\Phi(Z^0)}{\Phi(Z^0) - K^2(iK)^{1/3}},\ \Phi(Z^0) = \frac{d\mathrm{Ai}(Z^0)}{dZ}\left[\int_{Z^0}^{\infty}\mathrm{Ai}(Z)\,dZ\right]^{-1} \tag{4.8}$$

of the instantaneous streamline displacement.

5. In order to establish the basic characteristics of the velocity field at different oscillator frequencies, it is not necessary to know the details of the treatment of integral transforms (4.3), which can be understood with the help of techniques developed in [14]. After replacing the real quantities Ω and K in the expressions (4.6) for Z^0 by arbitrary complex quantities $-i\omega$ and $-ik$, respectively, the numerator of the right-hand side of the equations for $\overline{A}$ is equated to zero. It results in the dispersion equation (3.4), relating the frequency of free oscillations to the wave numbers. As mentioned above, these fluctuations are the limiting form of Tollmien–Schlichting waves (which corresponds to the Reynolds number tending to infinity). The fact that the first mode of free oscillations can be stable as well as unstable is of special significance. According to computations, the real part of ω for the first root ζ_1 of the dispersion equation (3.4) changes sign when it passes through zero with $k = \pm k_* i = \pm 1.0003i$. Conversion of results from [5, 6] shows that in this case $\omega = \pm\Omega_* i = \pm 2.298i$.

Since $\omega = i\Omega$, $k = ik$, the quantity $\Omega = \Omega_*$ when $K = -k_*$, as determined by the first root Z_1^0 of the dispersion equation. Consider now the inverse Fourier transform in the complex plane K, assuming the streamwise coordinate $x > 0$. In order to ensure the choice of single-valued branch of the function $K^{1/3}$ in the expression for Z, we introduce a cut along the positive imaginary axis (Fig. 2). Using Cauchy's theorem, the integrals along the real axis are given by the product of $2\pi i$ and the sum of all residues of the analytical functions within the integral and the integrals on either side of the cut. Of the infinite sum of residues we take only those that correspond to the first root Z_1^0 of the dispersion equation, where the frequency Ω is fixed, and K takes any complex value. Let K_1 be the value corresponding to the first root. Referring to Eq. (4.8), we conclude that the contribution to the expression for the instantaneous displacement of streamlines, given by the residues, equals

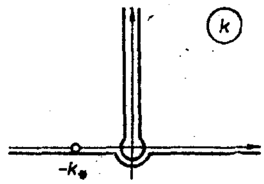

Fig. 2

$$\frac{3iK_1\bar{h}(K_1)\, I(Z_1^0)\,\Phi(Z_1^0)}{K_1^2 (iK_1)^{1/3}\left[2Z_1^0 \mathrm{Ai}(Z_1^0)+7I(Z_1^0)\right]+2(Z_1^0)^2\mathrm{Ai}(Z_1^0)}\, e^{iK_1x}, \quad I(Z_1^0)=\int_{Z_1^0}^{\infty}\mathrm{Ai}(Z)\,dZ.$$

Similar terms proportional to exp ik_1x appear in the equations for the remaining parameters of the fluid. It is clear that as $\Omega \to \Omega_* = 2.298$, we have Re $K_1 \to -k_* = -1.0003$, Im $K_1 \to 0$. Thus, disturbances generated by the oscillator decay at infinity downstream only for sufficiently low frequencies $\Omega < \Omega_*$, and, as Ω approaches Ω_*, the damping rate becomes sufficiently small. In the limit as $\Omega = \Omega_*$, disturbances become neutral with constant amplitude along the jet. However, if it is a high-frequency oscillator with $\Omega > \Omega_*$, disturbances should exponentially amplify downstream, as indicated by the above established characteristics of the jet stability.

6. Keeping in mind that the above-described results were obtained in a special dimensionless reference system, introduced by the Eqs. (2.1), (2.2), and (2.6), the converson of frequency ω and wave number k in this system to frequency ν and wave number α in the original (also dimensionless) system, is carried out with the help of the relations

$$\omega = \Delta^{2/7}\lambda^{-8/7}\mathrm{Re}^{-2/7}\nu, \quad k = \Delta^{3/7}\lambda^{-5/7}\mathrm{Re}^{-3/7}\alpha.$$

The substitution of these relations in the dispersion equation (3.4) makes it possible to express the constant as ζ and $\zeta = \lambda^{-2/3}\nu/\alpha^{2/3}$, and its right-hand side Q in the form $Q = \Delta\lambda^{-5/3}\mathrm{Re}^{-1}\alpha^{7/3}$.

It is clear from the above that the condition for the stability of freely interacting incompressible jet on a flat plate appears as

$$\Delta^{3/7}\lambda^{-5/7}\mathrm{Re}^{-3/7}|\alpha| \leqslant k_* = 1.0003. \tag{6.1}$$

A similar condition related to the disturbance frequency of the oscillator has the form

$$\Delta^{2/7}\lambda^{-8/7}\mathrm{Re}^{-2/7}|\nu| \leqslant \Omega_* = 2.298. \tag{6.2}$$

The sign of the equations in expressions (6.1) and (6.2) correspond to neutral long-wavelength disturbances whose amplitude remains constant in time and along the jet. The change in the position of the neutral curve in the Re–$|\alpha|$ plane generally used in linear stability theory, due to a variation in the coefficients Δ and λ, can be estimated with the above formulation. Since the first of these coefficients is determined by the velocity profile in the boundary layer, Eqs. (6.1), (6.2) make it possible to compare different fluid flows mentioned in [1-4].

There does not seem to be any experimental data on the stability of such flows. However, a large number of experiments have been conducted on the growth of disturbances in the boundary layer on a flat plate in a uniform flow [15, 16]. Experimental results agree very well with computations of the neutral curve using linear stability theory. On the other hand, experimental data indicate that the growth of unstable disturbances leads ultimately to the breakdown of laminar flow in the boundary layer and its transition to turbulence. Transition is preceded by a highly nonlinear process of amplification of disturbances in the basic Tollmien–Schlichting waves.

As shown in [5, 6], the asymptotic behavior of large wavelength disturbances with the critical layer of neutral disturbances lying close to the flat plate is correctly established with the help of free interaction theory, using Prandtl's boundary layer equations (2.3), with the self-induced pressure (2.4). This system was used in [17] to determine the limiting solutions branching off from the neutral stability curve, taking into consideration weakly nonlinear effects.

The complete set of boundary layer equations, with self-induced pressure, can be used to study appreciably nonlinear amplification of unstable Tollmien–Schlichting waves with large period. There is a fundamental interest in the understanding of the following problem: Do time-dependent solutions with stochastic properties exist for the above equations? If there is a positive solution, then Prandtl's boundary layer equation will be applicable to the study of the origin of turbulence.

LITERATURE CITED

1. E. Von Schmidt and W. Beckmann, "Das Temperatur-und Geschwindigkeitsfeld von einer Warme abgebenden senkrechten Platte bei naturliche Konvektion," Forsch. Gebiete Ingenieurwesens, Tech. Mech. Therm., 1, No. 10, 11 (1930).
2. T. Von Karman, "Über laminare und turbulente Reibung," Z. Angew. Math. Mech., 1, No. 4 (1921).
3. A. F. Messiter and A. Linan, "The vertical plate in laminar free convection; effects of leading and trailing edges and discontinuous temperature," Z. Angew. Math. Mech., 27, No. 5 (1976).
4. F. T. Smith and P. W. Duck, "Separation of jets or thermal boundary layers from a wall," Q. J. Mech. Appl. Math., 30, Pt. 2 (1977).
5. F. T. Smith, "On the nonparallel flow stability of the Blasius boundary layer," Proc. R. Soc. London, Ser. A., 366, No. 1724 (1979).
6. V. I. Zhuk and O. S. Ryzhov, "Free interaction and stability of incompressible boundary layer," Dokl. Akad. Nauk SSSR, 253, No. 6 (1980).
7. M. A. Gol'dshtik and V. N. Shtern, Hydrodynamic Stability and Turbulence [in Russian], Nauka, Novosibirsk (1977).
8. V. Ya. Neiland, "Asymptotic theory of viscous supersonic flows," Tr. Tsentr. Aerogidrodinam. Inst., 1524 (1974).
9. K. Stewartson, "Multistructured boundary layers on flat plates and related bodies," in: Advances in Applied Mechanics, Vol. 14, Academic Press, New York—London (1974).
10. W. Schneider, "Upstream propagation of unsteady disturbances in supersonic boundary layers," J. Fluid Mech., 63, Part 3 (1974).
11. S. N. Brown and P. G. Daniels, "On the viscous flow about the trailing edge of a rapidly oscillating plate," J. Fluid Mech., 67, Part 4 (1975).
12. O. S. Ryzhov and E. D. Terent'ev, "On nonstationary boundary layer with self-induced pressure," Prikl. Mat. Mekh., 41, No. 6 (1977).
13. O. S. Ryzhov and V. I. Zhuk, "Internal waves in the boundary layer with the self-induced pressure," J. Mecanique, 19, No. 3 (1980).
14. E. D. Terent'ev, "Computation of pressure using linear theory to study vibrations in supersonic boundary layer," Prikl. Mat. Mech., 43, No. 6 (1979).
15. H. L. Dryden, "Recent advances in the mechanics of boundary layer flow, in: Advances in Applied Mechanics, Vol. 1, Academic Press, New York (1948).
16. J. A. Ross, F. H. Barnes, et al., "The flat plate boundary layer, Pt. 3. Comparison of theory with experiment," J. Fluid Mech., 43, Pt. 4 (1970).
17. F. T. Smith, "Nonlinear stability of boundary layers for disturbances of various sizes," Proc. R. Soc. London, Ser. A., 368, No. 1735 (1979).